Non-Homothetic Demand and Income Divergence in a Ricardian Model of International Trade

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Abstract

We introduce a dynamic process of quality upgrading a là Grossman and Helpman (1991) into a Ricardian model of international trade with non-homothetic preferences. We show that in such a framework the income distribution across countries displays an inherent tendency towards divergence in the long run. The underlying reason for the divergence reminds us of the old Prebish (1950) and Singer (1950) hypothesis; namely, countries that specialise in goods with low income demand elasticity experience a secular decline in their terms of trade. Yet, in contrast with the previous literature in the subject, our model generates these dynamics purely from the *terms of trade effect*, in isolation from any other source of disparities. Furthermore, we show that the world economy undergoes an initial phase during which all countries display identical income, and inequality among them only arises once world productivity surpasses some minimum threshold.

1 Introduction

The main prediction of the Solow (1956) model, according to which poorer economies should catch up with richer ones, is rejected by the cross-country data – see, for example, Jones (1997) or Azariadis and Stachurski (2005). In the light of this evidence, the growth and development literature started looking for alternative theories which could lead to divergent dynamics or, at least, absence of convergence. Most of this literature initially abstracted from international trade and its effects on growth. However, the perception that international trade might exert differential impacts on countries depending on their relative level of development has been floating around since Prebish (1950) and Singer (1950). In their view, poorer economies specialise in commodities with low income demand elasticity. Hence, the story goes, as the world income rises demand deviates towards richer economies, improving their terms of trade and thereby magnifying initial income disparities.

Underlying the hypothesis of Prebish and Singer is the idea that individuals' preferences are non-homothetic. As a result, as people become wealthier expenditure shares bias towards luxury goods, which are produced by the industrialised economies. Following this line of reasoning, Matsuyama (2000) builds a Ricardian model with non-homothetic preferences based on a hierarchical ordering of needs that generates this pattern of international trade. Yet, neither in the writings by Prebish or Singer, nor in the model proposed by Matsuyama, any (endogenous) explanation is provided for why poorer countries should specialise in commodities with low income demand elasticity in the first place. In fact, in those articles the specific international trade pattern that arises is not the *cause* of disparities across countries, but actually the result of it (what trade does in those papers is magnifying those initial disparities, which are exogenously predetermined).

We propose an endogenous mechanism for determining the pattern of international specialisation in the production of commodities with different income elasticity of demand, based on a process of substitution to higher quality goods as wealth increases. The quality ladder structure in our paper is analogous to the one constructed by Grossman and Helpman (1991) – that is, in a continuum of horizontally differentiated commodities, an infinite number of qualities for each variety of good are available in the market. However, in contrast with Grossman and Helpman, in our framework the optimal expenditures shares do not remain constant across varieties as income rises. The idea is that, because the marginal valuation of quality of consumption increases with income (compared to that of physical consumption), as individuals become richer they optimally shift resources towards those commodities whose quality has risen more. The distribution of quality upgrades for different varieties is itself endogenous to the model. More precisely, the optimal increase in quality for each variety depends on the aggregate cost structure, and in particular on the cost of quality upgrading. Consequently, through the resulting process of quality upgrading and substitution, the underlying technological structure ultimately determines (*de facto*) which varieties of goods are luxuries and which necessities.

By introducing these dynamics of quality upgrading and demand determination into a Ricardian model of trade, we find that long-run income disparities among economies may arise as a consequence of heterogeneous cost structures for the quality upgrading. Specifically, we show that countries that specialise in producing varieties with high cost elasticity of quality upgrading (hence, optimally kept at a relatively lower level in the quality-ladder), experience a secular decline in their terms of trade, because these commodities display low income demand elasticity.

Interestingly, our model *endogenously* generates divergence in a world-economy without relying on any initial *absolute advantage* for any specific country, but only through the dynamics of the terms of trade. In particular, our world-economy experiences an initial phase in which all countries have identical income, and disparities eventually arise only once the world productivity surpasses some minimum threshold.[1] The most remarkable feature of this mechanism lies in that, even though comparative advantages are held fixed, there is no pre-ordered reason for considering any economy *initially* "better" than any other one. At any given level of wealth, it is only the distribution of the cost of quality upgrading across commodities what governs towards which goods, and hence towards which countries, aggregate demand will bias in the long run.

This mechanism represents the most distinctive feature of our model. The previous literature attempting to model a world economy with North-South trade has always needed to rely on some sort of absolute advantage by the North over the South. For example, Matsuyama (2000) and Flam and Helpman (1987) assume that the North is intrinsically more productive than the South at any variety of goods. Similarly, Stokey (1991) postulates that the stock of human capital in North is larger than in the South, turning thus the former more productive than the latter. In this sense, all these authors somehow assume that the North is initially richer than the South, and just then they let international trade play a role in the model, magnifying those initial

[1] This particular feature of the dynamics not only contrasts with previous literature in the subject –e.g. Flam and Helpman (1987), Stokey (1991), and Matsuyama (2000)–, but also seems to be in line with the world income distribution dynamics before and after the Industrial Revolution – see Galor (2005).

disparities.

In addition to fully link the world income distribution to the dynamics of international trade, we provide a novel characterisation for the consumer preferences which permits obtaining non-homotheticity in a very standard fashion, without imposing *ad hoc* assumptions on the commodity space. The former literature typically derives non-homotheticity in demand by assuming that goods are ordered along an exogenous hierarchical line of necessity. In particular, Matsuyama (2000) and Flam and Helpman (1987) set an upper bound on physical consumption, forcing individuals to increase consumption by purchasing lower-ranked/less-urgent goods as their income rises. A similar mechanism is also present in Stokey (1991) within a characteristics framework. She sets a upper bound on marginal utility for all characteristics, while lets the price of the goods grow unboundedly as they add more characteristics; in this way, she establishes a line of hierarchy that monotonically maps into the quality space, where goods of higher quality remain unconsumed until income rises sufficiently to make them affordable. In contrast, our model generates non-homotheticity in demand from an unrestricted standard consumer choice framework in which all commodities are in principle symmetric from the consumer viewpoint.[2]

The paper is organised as follows. Section 2 describes the set-up of the model. Section 3 solves the consumer's problem in a partial equilibrium set-up, illustrating the specificities of the non-homotheticity of demand in our model. Section 4 computes the general equilibrium in the world economy, examining the effects of uniform aggregate productivity growth and income inequality. Section 5 concludes. The appendices contain the ommitted proofs and some additional algebraic derivations used in the main text.

2 Structure of the Model

We consider a world composed by two countries: the *North* and the *South*. These two economies share a common commodity space, defined along two distinct dimensions: *horizontal* and *vertical*. The first dimension (*horizontal*) designates the *variety* (or *industry*) – e.g., food, TV, etc. Different varieties are indexed by the letter v along the variety space $\mathbb{V} \subset \mathbb{R} : v \in [0, 1]$. The second dimension (*vertical*) refers to the intrinsic *quality* of the good of a particular variety v – e.g., organic vs. non-organic food, LCD TV vs. cathode ray tube TV, etc. Within each

[2] It is worth noting that, although we could easily introduce some sort of hierarchical degree of need across commodities into the model, this would just add an additional (and orthogonal) source of non-homotheticity, which we prefer to assume away for the clarity of the exposition.

variety $v \in \mathbb{V}$, commodities are *vertically* ordered by the quality-index q belonging to the set $\mathbb{Q} \subset \mathbb{R} : q \in [1, \infty)$, where a higher q denotes a higher quality level. The commodity space is then given by the set $\mathbb{V} \times \mathbb{Q} = [0, 1] \times [1, \infty)$, and each commodity is identified by a pair $(v, q) \in \mathbb{V} \times \mathbb{Q}$.[3] We assume that all commodities are tradable, and there are no transport cost and no barriers or tariffs affecting international trade.

2.1 Preferences and Budget Constraint

Both the North and the South are inhabited by a continuum of individuals with identical preferences defined over the commodity space $\mathbb{V} \times \mathbb{Q}$. Hereafter, we adopt the following notation: unstarred latin symbols refer to the North, starred ones to the South.

Denote by $x_{vq} \in \mathbb{R}_+$ the consumed *quantity* of commodity $(v, q) \in \mathbb{V} \times \mathbb{Q}$ (i.e. the consumed quantity of variety v in quality q) by a representative individual from the North. His preferences are summarised by the following utility function:

$$U = \int_{\mathbb{V}} \ln \left[\int_{\mathbb{Q}} \max \{x_{vq}, (x_{vq})^q\} \, dq \right] dv \tag{1}$$

This utility function captures the notion that quality is a desirable feature in commodities. However, notice that according to specification in (1), although quality is never bad, it only *magnifies* the utility derived from (physical) consumption when $x_{vq} > 1$. This last property of (1) intends to capture the idea that individuals first seek to satisfy their basic consumption needs, and just after these basic needs are met, do they start paying attention to the quality dimension of the goods they consume.

Some additional properties about the utility function specified in (1) are worth noting. First, U is a sum in logs of aggregate quality-adjusted consumption across varieties; hence, for each variety v, marginal utility is unbounded above as consumption approaches zero, implying that all varieties will be actively consumed in an optimum (this result differs from Matsuyama, 2000). Second, convexity in quantities of the inner integrals of U means that individuals will optimally consume only *one* type of quality for each variety v. Third, considering two different levels of the quality-index $\underline{q} < \overline{q}$ for the same variety v, the marginal rate of substitution of $x_{v\overline{q}}$ for $x_{v\underline{q}}$

[3] In our setup, different varieties should be then understood as commodities that aim at satisfying different *needs*. On the other hand, different qualities for a particular variety refer to the *extent* (or *degree*) in which the *need* is actually satisfied by the commodity. In that regard, food satisfies a different need when compared to TVs (physiological nutrition vs. visual entertainment), but an LCD TV satisfies the need for visual entertainment (objectively!) *better* than a cathode ray tube TV.

is non-decreasing along a proportional *expansion path* of $x_{v\overline{q}}$ and $x_{v\underline{q}}$.[4] This last property of (1) is key in our paper in order to allow demand functions to display non-homothetic behaviour, where the rich spend a larger fraction of their income in high-qualities than the poor.

Each individual is endowed with one unit of *effective* labour. Labour is immobile across countries. As a result, each individual in the North supplies inelastically his entire labour endowment to Northern firms in return of a wage $w \in \mathbb{R}_{++}$ (hereafter, all prices are measured in a common *numeraire*). This wage represents the only source of income for the individual. Therefore, his budget constraint reads as follows:

$$\int_{\mathbb{V}} \int_{\mathbb{Q}} p_{vq}\, x_{vq}\, dq\, dv \leq w \tag{2}$$

where $p_{vq} \in \mathbb{R}_{++}$ denotes the (international) price of each unit of commodity $(v, q) \in \mathbb{V} \times \mathbb{Q}$.

We define $\beta_v \equiv w^{-1} \int_{\mathbb{Q}} p_{vq} x_{vq} dq$ as the *demand intensity* of variety $v \in \mathbb{V}$.[5] In the optimum, given the specification in (1), the budget constraint (2) will naturally bind. It is thus straightforward to notice that demand intensities will sum up to one across varieties (i.e. $\int_{\mathbb{V}} \beta_v dv = 1$).

All individuals in the world face the same prices for the reproducible commodities. As a result, the analogous expressions in (1) and (2) corresponding to the South read, respectively, as follows: $U^* = \int_{\mathbb{V}} \ln \left[\int_{\mathbb{Q}} \max \left\{ x^*_{vq}, \left(x^*_{vq}\right)^q \right\} dq \right] dv$ and $\int_{\mathbb{V}} \int_{\mathbb{Q}} p_{vq} x^*_{vq} dq dv \leq w^*$; where w^* denotes the wage in the South in terms of the common *numeraire* (clearly, since labour is immobile, w and w^* need not be equal).

2.2 Technology

In both countries competitive firms produce commodities based on linear production functions in which labour represents their only input. We let unit labour requirements vary both across varieties and across qualities of each variety. Also, we let unit labour requirements differ across countries. In particular, in the North the unit labour requirement for commodity $(v, q) \in \mathbb{V} \times \mathbb{Q}$ is given by $c_{vq} = a(v)\, q^{\eta(v)}/\kappa$, while in the South is given by $c^*_{vq} = a^*(v)\, q^{\eta(v)}/\kappa$; where $\kappa > 0$ denotes a world *aggregate-productivity* parameter, $a(v)$ and $a^*(v)$ represents *variety-specific*

[4] To see this, note the $MRS(x_{v\overline{q}}, x_{v\underline{q}})$ is defined by $(\partial U/\partial x_{v\overline{q}}) \big/ (\partial U/\partial x_{v\underline{q}})$, and along a proportional expansion path $x_{v\overline{q}} = k\, x_{v\underline{q}}$, where $k > 0$. Then, for $x_{v\underline{q}}, x_{v\overline{q}} > 1$:

$$MRS(k\, x_{v\underline{q}}, x_{v\underline{q}}) = \frac{\overline{q}}{\underline{q}} k^{\overline{q}-1} \left(x_{v\underline{q}}\right)^{\overline{q}-\underline{q}},$$

from where it is clear that, along the *ray* $x_{v\overline{q}} = k\, x_{v\underline{q}}$, $MRS(x_{v\overline{q}}, x_{v\underline{q}})$ is increasing in $x_{v\underline{q}}$.

[5] We borrow this nomenclature from Horvath (2000).

technological parameters which can differ in the North and the South, and $\eta(v)$ summarises the *cost elasticity of quality upgrading* for each variety v which is assumed to be the same for the North and the South. We suppose that $a(v) : [0,1] \rightarrow \mathbb{R}_{++}$, where $a'(\cdot) \geq 0$; analogously, $a^*(v) : [0,1] \rightarrow \mathbb{R}_{++}$, where $a^{*\prime}(\cdot) \geq 0$. We also assume that $\eta(v) : [0,1] \rightarrow \mathbb{R}_{++}$, where $h'(\cdot) > 0$ and $h(0) > 1$.[6]

The next assumption dictates the pattern of *comparative* advantages across countries.

Assumption 1 *Let* $A(v) \equiv a^*(v)/a(v)$. *We suppose:* $A(0.5) = 1$ *and* $A'(v) < 0$.

Assumption 1 is the only source of heterogeneity across countries. In particular, this last assumption implies that the North enjoys a comparative advantage in lower-indexed commodities, while the South has a comparative advantage in the upper-indexed commodities.

Notice that given the cost functions c_{vq} and c^*_{vq} specified above, we are allowing countries to possibly display identical income per-head in equilibrium, since we are not imposing any direct source of *absolute* advantage in the model. Furthermore, notice that because $\eta(v)$ is the same for the North and the South, the nature of the comparative advantages do not change as we move up in the quality-ladder. In that sense, the comparative advantages always refer to particular varieties of goods, irrespective of the quality at which this particular variety is actually produced (for example, a country that has a comparative advantage in producing foodstuff, will have this advantage in producing both organic and non-organic food products).

In our world economy, each country will naturally specialise in those commodities which they can produce more cheaply. As a result, the *international* price of each commodity will be given by $p_{vq} = \min\left\{c_{vq}w, c^*_{vq}w^*\right\}$. Given Assumptions 1, we can write the international price of each commodity $(v,q) \in \mathbb{V} \times \mathbb{Q}$ as follows:

$$p_{vq} = \kappa^{-1}\alpha(v)\, q^{\eta(v)}, \tag{3}$$

where $\alpha(v) \equiv \min\{a(v)\,w, a^*(v)\,w^*\}$. In addition, from (3) we can determine the marginal variety m (that is, the variety can be produced by both countries at the same cost) as:

$$w/w^* = A(m). \tag{4}$$

[6] From the labour requirements functions it is apparent that qualitative upgrade is costly, which seems a natural assumption to make. Additionally, from our assumptions it follows that $\eta(v) > 1$ for all $v \in \mathbb{V}$, which implies that the marginal cost of improving quality is, for each variety, increasing along the quality-space. In that sense, this assumption also seems quite natural, as it reflects the fact that subsequent quality improvements become increasingly costly. Finally, note that $\eta'(\cdot) > 0$ – coupled with $a'(\cdot) \geq 0$ – implies that varieties are sorted by their cost of quality upgrading.

Equation (4) implies that, given the relative wage w/w^*, the North will produce all the varieties in the interval $[0, m]$ and the South will produce all the varieties within $[m, 1]$.

3 The Individual's Optimal Consumption Choice

In this section we illustrate the optimal consumption choice of a representative individual from the North, given the set of prices in the world economy. The results so obtained can be easily extended to an individual from the South, which is done in Appendix B.

The individual in the North chooses the quantities $x_{vq} \in \mathbb{R}_+$ to consume of each commodity $(v, q) \in \mathbb{V} \times \mathbb{Q}$ in order to solve the following problem:

$$\begin{aligned} \max_{\{x_{vq}\}_{(v,q)\in\mathbb{V}\times\mathbb{Q}}} \quad & U = \int_{\mathbb{V}} \ln \left[\int_{\mathbb{Q}} \max\left\{x_{vq}, (x_{vq})^q\right\} dq \right] dv, \\ \text{subject to:} \quad & \textstyle\int_{\mathbb{V}} \beta_v dv = 1, \\ & p_{vq} = \kappa^{-1} \alpha(v)\, q^{\eta(v)},\ \forall (v, q) \in \mathbb{V} \times \mathbb{Q}. \end{aligned} \tag{5}$$

In order to solve (5), it proves convenient to state the following preliminary results.

Lemma 1 (Preliminary Results)

(i) For each variety $v \in \mathbb{V}$, at most one quality, denoted henceforth by $q_v \in \mathbb{Q}$, is consumed in strictly positive amount in an optimum; formally: $x_{vq_v} \geq 0$, $x_{vq} = 0$, $\forall q \neq q_v$.

(ii) Take the commodity x_{vq_v} for any variety $v \in \mathbb{V}$: if $q_v > 1$, then $x_{vq_v} > 1$.

Proof. See Appendix C. ■

From Lemma 1, Part (i), it immediately follows that the income devoted to purchasing commodities of variety v is entirely spent on quality q_v. Hence, for each $v \in \mathbb{V}$ the consumed quantity of the optimal quality q_v is given by $x_{vq_v} = \beta_v w / p_{vq_v}$. In addition, from Part (ii), it follows that we may replace the inner integral $\int_{\mathbb{Q}} \max\left\{x^*_{vq}, \left(x^*_{vq}\right)^q\right\} dq$ in (5) by the simpler expression $\int_{\mathbb{Q}} \left(x^*_{vq}\right)^q dq$, without altering any of the final results of that problem.[7]

Given Lemma 1 the individual's optimisation problem can be thus restated in a simpler form in terms of two sets of control variables replacing quantity x_{vq_v} for each variety $v \in$

[7] To this more clearly, notice that (keeping in mind the physical constraints) $x_{vq_v} < (x_{vq_v})^{q_v}$ if and only if $x_{vq_v} < 1$ and $q_v < 1$, which cannot hold true according to Lemma 1, Part (ii).

$\mathbb{V}$ In particular, an individual from the North then solves the following *reduced-form* of the optimisation problem:

$$\begin{aligned} \max_{\{q_v,\beta_v\}_{v\in\mathbb{V}}} \quad & U = \int_{\mathbb{V}} q_v \ln\left(\frac{w\beta_v}{p_{vq_v}}\right) dv, \\ \textit{subject to:} \quad & \int_{\mathbb{V}} \beta_v dv = 1, \\ & q_v \geq 1, \ \forall v \in \mathbb{V}, \\ & p_{vq_v} = \kappa^{-1}\alpha(v)\, q_v^{\eta(v)}, \ \forall v \in \mathbb{V}. \end{aligned} \tag{6}$$

The first-order conditions (6) are stated in the Appendix A. From those first-order conditions we may obtain the following expression for each β_v in the optimum:

$$\beta_v = \frac{q_v}{\int_{\mathbb{V}} q_z dz}, \ \forall v \in \mathbb{V}. \tag{7}$$

The denominator of the right-hand side of (7) can be regarded as an aggregate index measuring the optimal consumption bundle's *average quality*, and is henceforth denoted by $Q \equiv \int_{\mathbb{V}} q_z dz$. Notice that, according to (7), the fraction of income spent on variety v is determined by its optimal quality relative to the average quality of consumption. In that regard, if all varieties were optimally consumed at identical quality degrees (i.e. if $q_v = Q$, $\forall v \in \mathbb{V}$), then $\beta_v = 1$ would hold for all $v \in \mathbb{V}$, and the model would be as in Dornbusch, Fischer and Samuelson (1977).

3.1 Distribution of Qualities and Demand Intensities across Varieties

Given the technology in the world economy –summarised by κ, $\alpha(\cdot)$ and $\eta(\cdot)$– it is possible to characterise the distribution of the optimal qualities across varieties according to their position within the set $\mathbb{V}$. Lemma 2 provides the first result in that direction.

Lemma 2 *Consider two varieties* $\underline{v}, \bar{v} \in \mathbb{V}$*, such that* $\underline{v} < \bar{v}$*. Then:* $q_{\underline{v}} \geq q_{\bar{v}}$*, with strict inequality if and only if* $q_{\underline{v}} > 1$*.*

Proof. See Appendix C. ■

Lemma 2, implies that the consumed quality q_v is non-increasing in the variety-index v. The underlying intuition for Lemma 2 is straightforward – those varieties which can be more cheaply upgraded tend to be optimally consumed in higher quality levels.

The monotonicity of q_v implied by Lemma 2 allows us to split the variety-space in two disjoint subsets. The first subset containing varieties that are bound to be consumed at the baseline quality level (i.e. with $q_v = 1$) – these are the higher-indexed varieties. The second one comprising the varieties for which the constraint $q_v \geq 1$ does not bind – these are the lower-indexed varieties. Let us denote by $\mathbb{L} \subseteq \mathbb{V}$ the latter subset.

Definition 1 *Let* $\mathbb{L} = \{v \in \mathbb{V} : \lambda_v = 0\}$

Both $\mathbb{L} = \emptyset$ and $\mathbb{L} = \mathbb{V}$ are in principle possible. In fact, $\mathbb{L} = \emptyset$ will hold if κ is sufficiently small, while $\mathbb{L} = \mathbb{V}$ will hold if κ is sufficiently large. (See Lemma 3 ahead in the text).

Finally, regarding the optimal demand intensities, from the condition in (7) we can observe that, in the optimum, demand intensities are set proportional to the optimal qualities. As a result, the distribution of β_v across varieties will qualitatively mirror the one of q_v.

3.2 Effects of Aggregate Productivity Shocks on Demand

When the production technology is subject to changes, both *substitution-effects* (due to adjustments in relative prices) and *income-effects* (due to the overall effect of variations in productivity) arise. Here we focus our attention solely on income-effects. In order to isolate income-effects from substitution-effects, we let the parameter κ, while we keep constant the functions $a(\cdot)$, $a^*(\cdot)$ and $\eta(\cdot)$.

Lemma 3 *Let* $\underline{\kappa} \equiv a(0) \exp[\eta(0)]$*. Then:*
(i) for all $\kappa \in (0, \underline{\kappa}) : \mathbb{L} = \emptyset$*;*
(ii) for all $\kappa \geq \underline{\kappa} : \mathbb{L} = [0, \tilde{v}(\kappa)]$*, where* $\tilde{v}(\kappa) : [\underline{\kappa}, \infty) \rightarrow [0, 1]$*,* $\tilde{v}(\underline{\kappa}) = 0$*, and* $\tilde{v}'(\kappa) > 0$ *whenever* $\tilde{v}(\kappa) < 1$*.*

Proof. See Appendix C. ■

In short, Lemma 3 implies that the subset of varieties consumed at the *baseline* quality level initially comprises the entire set $\mathbb{V}$, and eventually starts narrowing as world aggregate productivity rises beyond the threshold $\underline{\kappa}$. The next lemma describes in further detail how optimal qualities evolve as the parameter κ increases.

Lemma 4

i) If $\kappa \in (0, \underline{\kappa})$*:* $\partial q_v / \partial \kappa = 0$ *for all* $v \in \mathbb{V}$*;*

ii) If $\kappa \geq \underline{\kappa}$*: a) for all* $v \in \mathbb{L}$*,* $\partial q_v / \partial \kappa > 0$*; b) for all* $v \notin \mathbb{L}$*,* $\partial q_v / \partial \kappa = 0$*; c) for all* $\underline{v}, \overline{v} \in \mathbb{V}$*, such that* $\underline{v} < \overline{v}$*,* $\partial q_{\underline{v}} / \partial \kappa \geq \partial q_{\overline{v}} / \partial \kappa$*, with strict inequality if and only if* $\underline{v} \in \mathbb{L}$*.*

Proof. See Appendix C. ■

Lemma 4 shows that, for all varieties belonging to $\mathbb{L}$, quality increases when the aggregate productivity in the world rises. Furthermore, this effect is stronger for those varieties whose quality can be more cheaply upgraded – i.e., those varieties carrying a lower $\eta(v)$. On the other hand, we can observe that the optimal quality of varieties that do not belong to $\mathbb{L}$ does not respond to changes in κ.

Based on Lemma 3 and Lemma 4, we can accordingly identify two distinct regimes depending on the level of κ that prevails. First, we refer to an economy such that $\kappa \leq \underline{\kappa}$ as a *subsistence economy* – in a subsistence economy all varieties are consumed at the baseline quality level. Second, we refer to an economy with $\kappa > \underline{\kappa}$ as a *modern economy* – in a modern economy some varieties (and possibly all of them) are consumed strictly above the baseline quality level. In what follows we proceed to further characterise these two regimes.

Subsistence Economy – $\kappa \leq \underline{\kappa}$

In this regime $q_v = 1$ holds for all $v \in \mathbb{V}$. This in turn means that $Q = 1$ and $\beta_v = 1$ must hold for all $v \in \mathbb{V}$ as well. Thus, in a subsistence economy demand intensities remain constant and equal to one for all varieties as κ increases.[8] In that regard, a subsistence economy displays analogous behaviour to the economy discussed in Dornbusch *et al* (1977), where demand intensities are homothetic across varieties.

Modern Economy – $\kappa > \underline{\kappa}$

This regime is characterised by $q_v > 1$ for all $v \in [0, \tilde{v}(\kappa))$. Hence, the average quality can be written as $Q = 1 - \tilde{v}(\kappa) + \int_0^{\tilde{v}(\kappa)} q_z \, dz$, from where it follows that $\partial Q/\partial\kappa = \int_0^{\tilde{v}(\kappa)} (\partial q_z/\partial\kappa) \, dz > 0$. Since $\partial q_v/\partial\kappa = 0$ for all $v \notin \mathbb{L}$, then $\partial\beta_v/\partial\kappa < 0$ must hold for all $v \notin \mathbb{L}$. As a result, given that $\int_{\mathbb{V}} \beta_v \, dv = 1$, it must thus be the case that the demand intensities of some (and possibly all) $v \in \mathbb{L}$ will increase as κ rises. Let $\mathbb{J} \subset \mathbb{V}$ denote the subset of $\mathbb{V}$ comprising all those varieties for which $\partial\beta_v/\partial\kappa > 0$.

Definition 2 *Let* $\mathbb{J} = \{v \in \mathbb{V} : \partial\beta_v/\partial\kappa > 0\}$.

In a subsistence economy $\mathbb{J} = \emptyset$, while in a modern economy $\mathbb{J} \neq \emptyset$. In other words, in a modern economy the homotheticity of demand intensities no longer holds, as a subset of

[8] It must be noted that this result applies only if $\kappa \leq \underline{\kappa}$ holds *after* performing the comparative statics exercise.

varieties whose income demand elasticity is larger than one shows up. Notice finally that $\mathbb{J} \subset \mathbb{L}$, since $\partial q_v / \partial \kappa > 0$ is a necessary condition for $\partial \beta_v / \partial \kappa > 0$ to hold.

The next proposition further characterises the behaviour of the demand intensity β_v of a generic variety v, in relation to those of the other varieties, as κ rises.

Proposition 1 *Consider any two varieties $\underline{v}, \overline{v} \in \mathbb{V}$, such that $\underline{v} < \overline{v}$. Then:*

i) If $\underline{v} \in \mathbb{J}$: $\partial \beta_{\underline{v}} / \partial \kappa > \partial \beta_{\overline{v}} / \partial \kappa$;

ii) If $\underline{v} \notin \mathbb{J}$: $\partial \beta_{\overline{v}} / \partial \kappa \leq 0$.

Proof. See Appendix C. ■

To interpret our previous results more clearly, notice that $\mathbb{J}$ may be understood as the set of *luxury goods*, where by luxury goods we refer to those varieties whose income demand elasticity is larger than 1. Since the set $\mathbb{J}$ comprises always lower-indexed varieties, the luxury goods are exactly those varieties whose quality degree q_v is relatively high compared to the average quality Q. In that sense, in our model it is the (relative) *quality* what determines whether or not a particular variety is *luxurious* and thus increases its share in the consumer's budget as he becomes richer. When individuals are still poor (i.e. when $\kappa \leq \underline{\kappa}$), satisfying all basic needs constitutes their main goal, leading them to keep the quality of all goods at the baseline level and setting accordingly equal shares for all varieties. As individuals become rich enough some (and eventually all) varieties start being consumed in higher quality degrees. In addition, the varieties whose quality degree is relatively higher attract increasingly larger income shares, as given the preference specification in (1) individuals tend to *value* high-quality commodities relatively more as they become wealthier.

From Proposition 1 we can derive the following corollary.

Corollary 1 *Let $\vartheta(v) \equiv \int_0^v \beta_z \, dz$. Then:*

(i) If $\kappa < \underline{\kappa} : \partial \vartheta(v) / \partial \kappa = 0$, $\forall \, v \in \mathbb{V}$;

(ii) If $\kappa \geq \underline{\kappa} : \partial \vartheta(v) / \partial \kappa > 0$, $\forall \, v \in [0, 1)$.

Proof. See Appendix C. ■

Corollary 1 somehow synthesizes the eventual non-homothetic behaviour of the demand schedules implied by our model. In particular, whenever $\kappa < \underline{\kappa}$, demand schedules are homothetic across varieties. However, when κ lies above the threshold $\underline{\kappa}$, income starts being spent in growing proportion on lower-indexed varieties.

4 General Equilibrium in the World Economy

In Section 3, we have studied the optimal consumption choice of an individual from the North, taking the wages in the North and in the South (i.e., w and w^*) as exogenously given. (In Appendix B, we do the same for the case of an individual from the South.) These wages in turn determine the prices of all reproducible commodities in the world economy through equation (3). Since the set of all relevant prices in the world economy have so far been exogenously set, our former analysis has described only partial equilibrium results.

The present section computes the general equilibrium in this world economy. This requires endogenising wages and, thereby, the prices of all reproducible commodities. Given that in a general equilibrium only relative prices are determined, we henceforth take the wage in South as the *numeraire*, by setting $w^* = 1$.

In order to disregard the effects of population heterogeneity across countries, we suppose that both the North and the South are inhabited by a continuum of individuals with identical mass, which we normalise to one.

A representative individual from the North will then solve:

$$\begin{aligned} \max_{\{q_v, \beta_v\}_{v\in\mathbb{V}}} \quad & U = \int_0^m q_v \ln\left(\frac{\beta_v \kappa}{a(v)\, q_v^{\eta(v)}}\right) dv + \int_m^1 q_v \ln\left(\frac{\beta_v \kappa\, w}{a^*(v)\, q_v^{\eta(v)}}\right) dv, \\ \text{subject to:} \quad & \int_{\mathbb{V}} \beta_v\, dv = 1;\ \text{and } q_v \geq 1,\ \forall v \in \mathbb{V}. \end{aligned} \tag{8}$$

On the other hand, a representative individual from the South solves:

$$\begin{aligned} \max_{\{q_v^*, \beta_v^*\}_{v\in\mathbb{V}}} \quad & U^* = \int_0^m q_v^* \ln\left(\frac{\beta_v^* \kappa}{a(v)\, q_v^{\eta(v)} w}\right) dv + \int_m^1 q_v^* \ln\left(\frac{\beta_v^* \kappa}{a^*(v)\, q_v^{\eta(v)}}\right) dv, \\ \text{subject to:} \quad & \int_{\mathbb{V}} \beta_v^*\, dv = 1;\ \text{and } q_v^* \geq 1,\ \forall v \in \mathbb{V}. \end{aligned} \tag{9}$$

The solution of (8) and (9) yields the demand functions of each variety $v \in \mathbb{V}$ for the North and the South, respectively. By using $\vartheta(v) \equiv \int_0^v \beta_z\, dz$ –as defined in Corollary 1– and $\vartheta^*(v) \equiv \int_0^v \beta_z^* dz$ –see Corollary 1 (South) in Appendix B–, we can write the equilibrium condition for the market of goods produced in the North as follows:

$$\vartheta(m)\, w + \vartheta^*(m) = w. \tag{10}$$

Condition (10) essentially says that the aggregate amount of income spent by the world in goods produced in the North must be equal to the aggregate income of the North. This condition can also be understood as the equilibrium condition for the labour market in the North.[9]

The world economy general equilibrium is determined by (4), (8), (9), and (10). We will henceforth focus our attention on the equilibrium values of w and m, and to how these two variables respond to some simple comparative statics exercises.

4.1 Worldwide Uniform Aggregate Productivity Growth

In this subsection, we look at the impact of changes in κ on the equilibrium values of w and m. We can split the results in two different cases.

Subsistence economies – $\kappa \leq \underline{\kappa}$

From our previous discussion, we can observe that when $\kappa \leq \underline{\kappa}$, the optimal demand intensities are set at $\beta_v = \beta_v^* = 1$ for all $v \in \mathbb{V}$. This result in turn implies that $\vartheta(m) = \vartheta^*(m) = m$. Therefore, (10) simplifies to:

$$w = m/(1-m)\,. \tag{11}$$

Combining then (4) with (11), leads to $m/(1-m) = A(m)$, from where it follows that, for all $\kappa \leq \underline{\kappa}$, in equilibrium: $w = 1$ and $m = 0.5$. That is, North and South exhibit equal income, and the pattern of regional specialisation is accordingly dictated by the "natural" comparative advantage of each country without relative-wage bias (i.e., the comparative advantage that derives purely from the heterogeneity in the technological structure implied by Assumption 1).[10]

Modern economies – $\kappa > \underline{\kappa}$

When aggregate productivity is sufficiently high, the income equality between the North and the South no longer holds. In particular, as κ rises above the threshold $\underline{\kappa}$, the terms of trade start moving in favour of the North, and thus the North becomes relatively richer than the South. Furthermore, the income disparity between the North and the South increases as κ keeps rising.

Proposition 2 *Suppose Assumptions 1 holds. In addition, suppose $\kappa > \underline{\kappa}$. Then, in equilibrium:*

[9] Because of the Walras' Law, an analogous condition can be derived for the equilibrium in the labour market in the South.

[10] Notice that, since $w = 1$ for all $\kappa \leq \underline{\kappa}$, in fact $\underline{\kappa} = \underline{\kappa}^*$ (that is, the threshold on κ that divides a subsistence-economy from a modern economy happens to be the same for the North and for the South). As a consequence, we can refer to *both* thresholds simply as $\underline{\kappa}$.

(i) $w > 1$ *and* $m < 0.5$.

(ii) $\partial w/\partial\kappa > 0$ *and* $\partial m/\partial\kappa < 0$.

Proof. Part (i). When $\kappa > \underline{\kappa}$, from Corollary 1 it follows that $\vartheta(m) > m$ and $\vartheta^*(m) > m$. As a result, by using (10), we can obtain:

$$w = \frac{\vartheta^*(m)}{1 - \vartheta(m)} > \frac{m}{1 - m}. \tag{12}$$

Combining next (12) with (4), and recalling Assumption 1 leads to:

$$A(m) = \frac{\vartheta^*(m)}{1 - \vartheta(m)} > \frac{m}{1 - m} \quad \Leftrightarrow \quad m < 0.5.$$

Finally, since $m < 0.5$, (4) implies that $w > 1$.

Part (ii). Next, to study how w and m vary as κ keeps rising above $\underline{\kappa}$, we differentiate the equilibrium conditions (4) and (10). This leads to:

$$\frac{\partial w}{\partial \kappa} = A'(m)\frac{\partial m}{\partial \kappa} \tag{13}$$

and

$$(w\beta_m + \beta_m^*)\frac{\partial m}{\partial \kappa} + \left(w\frac{\partial \vartheta(m)}{\partial w} + \vartheta(m) + \frac{\partial \vartheta^*(m)}{\partial w}\right)\frac{\partial w}{\partial \kappa} + \left(\frac{\partial \vartheta(m)}{\partial \kappa} + \frac{\partial \vartheta^*(m)}{\partial \kappa}\right) = \frac{\partial w}{\partial \kappa}, \tag{14}$$

where the first term in (14) uses the fact that $\partial\vartheta(m)/\partial m = \beta_m$ and $\partial\vartheta^*(m)/\partial m = \beta_m^*$. Plugging (13) into (14), we can obtain:

$$\frac{\partial m}{\partial \kappa} = \frac{\partial\vartheta(m)/\partial\kappa + \partial\vartheta^*(m)/\partial\kappa}{[1 - \vartheta(m) - w\,\partial\vartheta(m)/\partial w - \partial\vartheta^*(m)/\partial w]\,A'(m) - (w\beta_m + \beta_m^*)}. \tag{15}$$

For determining the sign of (15), we can use the following two results: first, Corollary 1 states that both $\partial\vartheta(m)/\partial\kappa > 0$ and $\partial\vartheta^*(m)/\partial\kappa > 0$; second, as shown in Appendix D, both $\partial\vartheta(m)/\partial w \leq 0$ and $\partial\vartheta^*(m)/\partial w \leq 0$. Therefore, since $1 - \vartheta(m) > 0$ and $A'(m) < 0$, then $\partial m/\partial\kappa < 0$ obtains from the right-hand side of (15). Finally, from (13) it then follows that $\partial w/\partial\kappa > 0$. ■

Proposition 2 shows that as the worldwide productivity parameter, κ, increases, the income in the North eventually begins diverging away from the income in the South. The reason for this income-divergence is that the North enjoys a comparative advantage in producing those commodities whose income-elasticity is relatively larger. As a consequence, as the world economy grows uniformly above $\underline{\kappa}$, aggregate world expenditure shifts towards the set of commodities produced by the North. The ensuing excess demand for commodities produced in the North causes excess labour demand in the North and w thus goes up. In turn, as w rises, the marginal variety moves to the left (i.e., m falls), and some of the varieties that used to be produced by the North start being now produced by the South, restoring the equilibrium in the labour markets.

4.2 Heterogeneous Population: the effects of income inequality

So far we have assumed that within each country all individuals are identical regarding all relevant features for our model. In this section we examine the general equilibrium consequences of introducing some degree of income heterogeneity within countries.

To keep the analysis short and concise, we introduce income inequality only in the South, while we maintain the assumption that the population in the North is homogeneous. In particular, we assume that the South is inhabited by two types of individuals: p and r, where the p stands for *poor* and r stands for *rich*. Each sub-group of Southerners has mass equal to 0.5. A poor Southerner is endowed with $1 - \iota$ units of effective labour, while a rich one is endowed with $1 + \iota$ units of it; where $\iota \in (0, 1)$. In the North everyone is endowed with 1 unit of effective labour.

Introducing income inequality in the South leads to interesting results when the poor Southerners are so poor that, in equilibrium, they consume all varieties at the baseline quality level, whereas in contrast rich Southerners afford to consume some of the varieties strictly above that level. To focus on such case, we accordingly assume that $\kappa = \underline{\kappa}$.

Proposition 3 *Suppose the population in the South is split in two groups, p and r, of equal mass; individuals in p are endowed with $1 - \iota$ units of effective labour and individuals in r are endowed with $1 + \iota$ units of it, where $\iota > 0$. Additionally, suppose that $\kappa = \underline{\kappa}$. Then, in equilibrium:*

(i) $w > 1$ and $m < 0.5$.

(ii) $\partial w / \partial \iota > 0$ and $\partial m / \partial \iota < 0$.

Proof. Part (i). When $\kappa = \underline{\kappa}$, in the South, $\vartheta_p^*(m) = m$ and $\vartheta_r^*(m) > m$; where $\vartheta_j^*(m)$ denotes the fraction of income that types $j \in \{p, r\}$ spend on varieties belonging to $[0, m)$. On the other hand, in the North, $\vartheta(m) = m$; since when $\kappa = \underline{\kappa}$ all Northerners optimally consume all varieties at the baseline level.[11] As a result, the equilibrium condition in the Northern labour market

[11] Recall from Lemma 3 that $\underline{\kappa} \equiv a(0) \exp[\eta(0)]$, which is independent of w/w^*.

(10) reads as follows:

$$m + \tfrac{1}{2}\left[(1-\iota)\, m + (1+\iota)\vartheta_r^*(m)\right] = w. \tag{16}$$

From (16), since $\vartheta_r^*(m) > m$, it immediately follows that $w > 1$, which in turn implies $m < 0.5$.

Part (ii). Totally differentiating (16), and using the fact that in equilibrium $\partial w/\partial\iota = A'(m)\,(\partial m/\partial\iota)$ must be verified, leads to:

$$\frac{\partial w}{\partial \iota} = \frac{(1+\iota)\,(\partial\vartheta_r^*/\partial w) + (\vartheta_r^*(m) - m)}{2(1-m) - (1+\iota)\,(\partial\vartheta_r^*/\partial w) - \left[2w + (1-\iota) + (1+\iota)\beta_{r,m}^*\right]/A'(m)} > 0. \tag{17}$$

The positive sign in (17) stems from the fact that $\partial\vartheta_r^*(m)/\partial\iota > 0$, $\partial\vartheta_r^*(m)/\partial w < 0$, and $A'(m) < 0$. ■

When $\kappa = \underline{\kappa}$, introducing income inequality in the South raises the relative wage in the North. This result is owing to the non-homotheticity of the demand schedules of the rich Southerners. More precisely, increasing ι transfers income from the poor Southerners who spend a fraction m of it in Northern goods, to the richer Southerners who spend a fraction $\vartheta_r^*(m) > m$ of their income on those commodities. As a result, aggregate demand for Northern goods goes up leading to higher w.

Incorporating inequality in the North in an analogous manner would carry similar consequences on w and m, as the richer Northerners would also tend to shift aggregate demand towards the goods produced in the North.

5 Conclusion

(COMING SOON)

Appendices

A First-Order Conditions for Consumption Choice in the North

The optimisation problem in (6) yields the following first-order conditions (where μ represents the Lagrange multiplier associated to the budget constraint and $\{\lambda_v\}_{v\in\mathbb{V}}$ denote those associated to the constraints $\{q_v \geq 1\}_{v\in\mathbb{V}}$):

$$\ln\left(\frac{\beta_v w}{\kappa^{-1}\alpha(v)\, q_v^{\eta(v)}}\right) - \eta(v) + \lambda_v = 0,\ \forall v \in \mathbb{V}; \tag{18}$$

$$\frac{q_v}{\beta_v} - \mu = 0,\ \forall v \in \mathbb{V}; \tag{19}$$

$$q_v - 1 \geq 0,\ \lambda_v \geq 0,\ \text{and}\ \lambda_v (q_v - 1) = 0,\ \forall v \in \mathbb{V}; \tag{20}$$

$$1 - \int_{\mathbb{V}} \beta_v\, dv = 0. \tag{21}$$

From (19), it follows that $\beta_v = q_v/\mu$. Then, replacing this last expression into (21) leads to $\int_{\mathbb{V}} q_z\, dz = \mu$, from where the condition (7) immediately obtains by using again (19).

By using the condition (7), we can rewrite (18) as:

$$\lambda_v = \eta(v) + \ln[\alpha(v)/w] - \ln\kappa + \ln Q + [\eta(v) - 1]\ln q_v. \tag{22}$$

The expression in (22) will be used in many of the following proofs.

B Optimal Consumption Choice in the South

Bearing in mind Assumption 1, we can write down the optimisation problem faced by a representative individual from the South as follows:

$$\max_{\{x^*_{vq}\}_{(v,q)\in\mathbb{V}\times\mathbb{Q}}} \quad U^* = \int_{\mathbb{V}} \ln\left[\int_{\mathbb{Q}} \max\left\{x^*_{vq}, \left(x^*_{vq}\right)^q\right\} dq\right] dv,$$

$$\text{subject to:} \quad \int_{\mathbb{V}} \beta^*_v dv = 1,$$

$$p_{vq} = \kappa^{-1} q^{\eta(v)} \alpha(v)\,,\ \forall (v,q) \in \mathbb{V}\times\mathbb{Q}.$$

Lemma 1 holds for x^*_{vq} in a similar fashion as for x_{vq}. Hence, we can re-state the problem specified above in terms of q^*_v and β^*_v, as it was previously done for the North (where q^*_v now denotes the quality of variety v consumed, in the optimum, in the South). This way, we can obtain the following first-order conditions, which constitute the analogous versions for the South of (7) and (22), respectively:

$$\beta^*_v = \frac{q^*_v}{\int_{\mathbb{V}} q^*_z\, dz},\ \forall v \in \mathbb{V}, \tag{23}$$

$$\lambda^*_v = \eta(v) + \ln[\alpha(v)/w^*] - \ln\kappa + \ln Q^* + [\eta(v) - 1]\ln q^*_v. \tag{24}$$

Given the first-order conditions in (23) and (24), all the ensuing results found in Section 3 follow through in qualitative terms. In particular, we can derive functions $\{q^*_v\}_{v\in\mathbb{V}}$ and $\{\beta^*_v\}_{v\in\mathbb{V}}$

displaying identical qualitative properties as their *counterparts* in the North, that is $\{q_v\}_{v\in\mathbb{V}}$ and $\{\beta_v\}_{v\in\mathbb{V}}$, in terms of Lemmas 2 - 4 and Porposition 1. Furthermore, we can similarly find the threshold $\underline{\kappa}^*$ for the worldwide aggregate-productivity parameter, which splits the South in the regimes of *subsistence-economy* and *modern-economy*; both exhibiting analogous properties as described for the North.[12] Finally, likewise for the North in Corollary 1, for the South the following holds:

Corollary 1 (South) Let $\vartheta^*(v) \equiv \int_0^v \beta_z^*\, dz$. Then:
(i) If $\kappa < \underline{\kappa}^* : \partial\vartheta^*(v)/\partial\kappa = 0, \forall\, v \in \mathbb{V}$;
(ii) If $\kappa \geq \underline{\kappa}^* : \partial\vartheta^*(v)/\partial\kappa > 0, \forall\, v \in [0,1)$.

C Omitted Proofs

Proof of Lemma 1.

Part (i). First, notice that utility is given by an additive function over logarithms. Optimization can thus be split in two stages: (a) maximise U with respect to the logarithms; (b) maximise those logarithms with respect to x_{vq}. About (b), notice that the logarithms are defined on the integral over convex functions of x_{vq}, and therefore are themselves convex functions. It follows that (b) optimally requires corner solutions, so the result claimed obtains.

Part (ii). The proof follows immediately from noting that, for all $v \in \mathbb{V}$, utility derived from consuming $x_{vq} \in (0,1]$ is independent of the quality-index q, while according to (3) the price of commodity $(v,q) \in \mathbb{V}\times\mathbb{Q}$ is strictly increasing along the quality space. ■

Proof of Lemma 2.

First, suppose $q_{\underline{v}} < q_{\overline{v}}$. Since $q_{\underline{v}} \geq 1$, then $q_{\overline{v}} > 1$, hence (22) paired with (20) yield: $\eta(\underline{v}) + \ln[\alpha(\underline{v})/w] - \ln(\kappa/Q) \geq 0$, while $\eta(\overline{v}) + \ln[\alpha(\overline{v})/w] - \ln(\kappa/Q) + [\eta(\overline{v}) - 1]\ln q_{\overline{v}} = 0$. Thus:

$$\eta(\underline{v}) + \ln\alpha(\underline{v}) \geq \eta(\overline{v}) + \ln\alpha(\overline{v}) + [\eta(\overline{v}) - 1]\ln q_{\overline{v}}.$$

This last equality in turn leads to:

$$[\eta(\overline{v}) - \eta(\underline{v})] + \ln[\alpha(\overline{v})/\alpha(\underline{v})] + [\eta(\overline{v}) - 1]\ln q_{\overline{v}} \leq 0,$$

which cannot possibly hold if $q_{\overline{v}} > 1$, as its left-hand side would then be strictly positive. Therefore, it must necessarily be the case that $q_{\underline{v}} \geq q_{\overline{v}}$.

[12] From Section 4, it is straightforward to observe that, given Assumption 1, $\underline{\kappa}^* = \underline{\kappa}$.

Next, suppose $q_{\underline{v}} = q_{\overline{v}} > 1$. In this case, (22) in conjunction with (20) yield:

$$\eta\left(\underline{v}\right) + \ln\alpha\left(\underline{v}\right) + \left[\eta\left(\underline{v}\right) - 1\right]\ln q_{\overline{v}} = \eta\left(\overline{v}\right) + \ln\alpha\left(\overline{v}\right) + \left[\eta\left(\overline{v}\right) - 1\right]\ln q_{\overline{v}} = 0.$$

This last equality in turn leads to:

$$-\left[\eta\left(\overline{v}\right) - \eta\left(\underline{v}\right)\right]\left(1 + \ln q_{\overline{v}}\right) = \ln\left[\alpha\left(\overline{v}\right)/\alpha\left(\underline{v}\right)\right].$$

However, this last equality cannot possibly hold since its right-hand side is strictly positive, while the left-hand side is negative. As a result, $q_{\underline{v}} > q_{\overline{v}}$ must necessarily hold when $q_{\underline{v}} > 1$. ■

Proof of Lemma 3.

In order to prove this lemma it proves convenient to state first the following result:

Claim 1 The optimal quality q_v of any variety $v \in \mathbb{V}$ can be written as follows:

$$q_v = \max\left\{\Phi_{0,v}\left(q_0\right)^{\Upsilon_{0,v}}, 1\right\}; \tag{25}$$

where:

$$\Phi_{0,v} \equiv \left[\frac{e^{\eta(0)}\alpha\left(0\right)}{e^{\eta(v)}\alpha\left(v\right)}\right]^{\frac{1}{\eta(v)-1}} > 0, \text{ and } \Upsilon_{0,v} \equiv \frac{\eta\left(0\right) - 1}{\eta\left(v\right) - 1} > 0.$$

Proof. See Appendix D.

Next, notice that, from (25), $\partial\Phi_{0,v}\left(v\right)/\partial v < 0$ and $\partial\Upsilon_{0,v}\left(v\right)/\partial v < 0$ since $a'\left(v\right) \geq 0$ and $\eta'\left(v\right) > 0$, hence the set $\mathbb{L} \subseteq \mathbb{V}$ comprises the lower-indexed varieties in $\mathbb{V}$, with $\tilde{v}(\underline{\kappa})$ representing its upper bound.

Part (i). When $\kappa \in (0, \underline{\kappa})$, conditions stipulated in (20) and (22) applied on $v = 0$ entail that: $q_0 = 1$ and $\lambda_0 > 0$. As a result, from Lemma 2 it follows that $q_v = 1$, $\forall v \in \mathbb{V}$. Therefore, since $a'\left(v\right) \geq 0$ and $\eta'\left(v\right) > 0$, again from (22), $\lambda_v > 0$ for all $v \in \mathbb{V}$ obtains, and thus $\mathbb{L} = \emptyset$.

Part (ii). Note that (22) applied on $v = 0$, in conjunction Lemma 2, implies that when $\kappa = \underline{\kappa}$, then $\lambda_0 = 0$ and $q_0 = 1$. Then, Lemma 2 implies $Q = 1$. Using these results in (22) yields: $\lambda_v = \eta\left(v\right) + \ln\left[\alpha\left(v\right)/w\right] - \ln\underline{\kappa}$, implying that $\lambda_v > 0$ for all $v \in (0, 1]$. As a result, the set $\mathbb{L} = 0$, meaning that $\tilde{v}(\underline{\kappa}) = 0$.

Claim 2 If $\tilde{v}(\kappa) < 1$, then $q_{\tilde{v}(\kappa)} = 1$.
Proof. See Appendix D.

Given Claim 2 and Lemma 2, the aggregate quality index can be written as follows: $Q = 1 - \tilde{v}(\kappa) + \int_0^{\tilde{v}(\kappa)} q_v\, dv$. Furthermore, observe that, whenever $\tilde{v}(\kappa) < 1$, $\ln\left(\kappa/Q\right) = \eta\left(\tilde{v}(\kappa)\right) +$

$\ln\left[\alpha\left(\tilde{v}(\kappa)\right)/w\right]$ must hold in equilibrium. This last condition yields, after some simple algebra, $Q = \kappa w \exp\left[-\eta\left(\tilde{v}\right)\right]/\alpha\left(\tilde{v}\right)$. In addition to that, because of Lemma 2, in equilibrium, $\left[\eta\left(v\right)-1\right]\ln q_v = \ln\left(\kappa/Q\right) - \eta\left(v\right) - \ln\left[\alpha\left(v\right)/w\right]$ must hold for any $v \leq \tilde{v}(\kappa)$. By using the former in the latter, after some algebra, we may obtain:

$$q_v = q_v\left(\tilde{v}(\kappa)\right) \equiv \left[\frac{\alpha\left(\tilde{v}(\kappa)\right)}{\alpha\left(v\right)}\right]^{\frac{1}{\eta(v)-1}} \exp\left[\frac{\eta\left(\tilde{v}(\kappa)\right)-\eta\left(v\right)}{\eta\left(v\right)-1}\right], \;\forall\, v \in [0, \tilde{v}(\kappa)]. \tag{26}$$

In equilibrium, it must be the case that:

$$\kappa w \exp\left[-\eta\left(\tilde{v}(\kappa)\right)\right]/a\left(\tilde{v}(\kappa)\right) = 1 - \tilde{v}(\kappa) + \int_0^{\tilde{v}(\kappa)} q_v\left(\tilde{v}(\kappa)\right)\,dv, \tag{27}$$

where the right hand-side of (27) uses (26). Computing the total differentiation of (27), yields after some algebra:[13]

$$\frac{Q}{\kappa}d\kappa = \left[\frac{\alpha'\left(\tilde{v}(\kappa)\right)}{\alpha\left(\tilde{v}(\kappa)\right)} + \eta'\left(\tilde{v}(\kappa)\right)\right]\left[Q + \int_0^{\tilde{v}(\kappa)} \frac{q_v}{\eta\left(v\right)-1}dv\right] d\tilde{v},$$

leading finally to:

$$\frac{d\tilde{v}}{d\kappa} = \left[\frac{\kappa}{Q}\left(\frac{\alpha'\left(\tilde{v}(\kappa)\right)}{\alpha\left(\tilde{v}(\kappa)\right)} + \eta'\left(\tilde{v}(\kappa)\right)\right)\left(1 - \tilde{v}(\kappa) + \int_0^{\tilde{v}(\kappa)} \frac{\eta\left(v\right)}{\eta\left(v\right)-1}q_v dv\right)\right]^{-1} > 0.$$

where the last inequality follows from the properties of the functions $\alpha\left(\cdot\right)$ and $\eta\left(\cdot\right)$ ■

Proof of Lemma 4.

Part (i). Proof follows immediately from noting that Lemma 3 implies that, whenever $\kappa \in (0, \underline{\kappa})$, $q_v = 1$ must hold for all $v \in \mathbb{V}$. Thus, whenever $\kappa \in (0, \underline{\kappa})$, $\partial q_v/\partial\kappa = 0$ for all $v \in \mathbb{V}$.

Part (ii.a). Differentiating (25), computed for any $v \in \mathbb{L}$, with respect to κ yields:

$$\frac{dq_v}{d\kappa} = \frac{\eta\left(0\right)-1}{\eta\left(v\right)-1}\left[\frac{e^{\eta(0)}\alpha\left(0\right)}{e^{\eta(v)}\alpha\left(v\right)}\right]^{\frac{1}{\eta(v)-1}} \left(q_0\right)^{\frac{\eta(0)-\eta(v)}{\eta(v)-1}} \frac{dq_0}{d\kappa}, \;\forall v \in \mathbb{L}.$$

Using again (25), the equation above can be written:

$$\frac{dq_v}{d\kappa} = \frac{\eta\left(0\right)-1}{\eta\left(v\right)-1}\frac{q_v}{q_0}\frac{dq_0}{d\kappa}, \;\forall v \in \mathbb{L}. \tag{28}$$

[13] One subtle caveat applies here. Even if both $wa(v)$ and $w^*a^*(v)$ are differentiable functions over the whole domain of v, the envelope function $\alpha(v)$ will not necessarily be so. In particular, $\alpha(v)$ may not be differentiable at the point $v = m$. As a result, if $\tilde{v} = m$, $\alpha'(\tilde{v})$ may not exist. In the very specific case where this "anomaly" holds, we take that $\alpha'(v) = \lim_{\triangle v \to 0^+} \frac{\triangle\alpha(v)}{\triangle v}$.

(Since $\eta(\cdot) > 1$, notice that $dq_v/d\kappa$ and $dq_0/d\kappa$ must then share the same sign, for all $v \in \mathbb{L}$). Given that $Q = 1 - \tilde{v}(\kappa) + \int_0^{\tilde{v}(\kappa)} q_z\, dz$, it follows that:

$$\frac{dQ}{d\kappa} = \int_0^{\tilde{v}(\kappa)} \frac{dq_z}{d\kappa} dz = \frac{1}{q_0} \left(\int_0^{\tilde{v}(\kappa)} \frac{\eta(0) - 1}{\eta(z) - 1} q_z dz \right) \frac{dq_0}{d\kappa}.$$

Applying (22) to $v = 0$ when $\lambda_0 = 0$ yields: $q_0 = \left[a(0) e^{\eta(0)} Q\right]^{-\frac{1}{\eta(0)-1}} \kappa^{\frac{1}{\eta(0)-1}}$. Thus:

$$\frac{dq_0}{d\kappa} = \frac{q_0}{\eta(0) - 1} \frac{Q}{\kappa} \left(1 - \tilde{v} + \int_0^{\tilde{v}(\kappa)} \frac{\eta(z)}{\eta(z) - 1} q_z\, dz \right)^{-1} > 0.$$

Therefore, from (28) it follows that $dq_v/d\kappa > 0$, $\forall v \in \mathbb{L}$ must also hold.

Part (ii.b). Since $q_v = 1$ must hold for all $v \notin \mathbb{L}$. Proof is analogous to that of Part (i) of this Proposition.

Part (ii.c). Part (ii.a) and (ii.b) of this Proposition, taken together, imply that $dq_{\underline{v}}/d\kappa = dq_{\overline{v}}/d\kappa = 0$ if $\underline{v}, \overline{v} \notin \mathbb{L}$, and $dq_{\underline{v}}/d\kappa > dq_{\overline{v}}/d\kappa = 0$ if $\underline{v} \in \mathbb{L}$ and $\overline{v} \notin \mathbb{L}$. For $\underline{v}, \overline{v} \in \mathbb{L}$, such that $\underline{v} < \overline{v}$, (28) leads to:

$$\frac{dq_{\underline{v}}}{d\kappa} = \frac{\eta(0) - 1}{\eta(\underline{v}) - 1} \frac{q_{\underline{v}}}{q_0} \frac{dq_0}{d\kappa} > \frac{\eta(0) - 1}{\eta(\overline{v}) - 1} \frac{q_{\overline{v}}}{q_0} \frac{dq_0}{d\kappa} = \frac{dq_{\overline{v}}}{d\kappa},$$

since by assumption $\eta(\underline{v}) < \eta(\overline{v})$ and, from Lemma 2, $q_{\underline{v}} > q_{\overline{v}}$. ■

Proof of Proposition 1.

Firstly, considering the definition of average quality, taking logarithms and differentiating (7) with respect to κ yields: $(d\beta_v/d\kappa)/\beta_v = (dq_v/d\kappa)/q_v - (dQ/d\kappa)/Q$. Using (28), we can write:

$$\frac{dq_{\underline{v}}}{d\kappa} \frac{1}{q_{\underline{v}}} = \frac{\eta(0) - 1}{\eta(\underline{v}) - 1} \frac{dq_0}{d\kappa} \frac{1}{q_0} > \frac{\eta(0) - 1}{\eta(\overline{v}) - 1} \frac{dq_0}{d\kappa} \frac{1}{q_0} = \frac{dq_{\overline{v}}}{d\kappa} \frac{1}{q_{\overline{v}}}.$$

Hence:

$$\frac{d\beta_{\underline{v}}}{d\kappa} \frac{1}{\beta_{\underline{v}}} > \frac{d\beta_{\overline{v}}}{d\kappa} \frac{1}{\beta_{\overline{v}}}. \tag{29}$$

Part (i). Using (29), the claim trivially follows by noting that, from Lemma 2 in conjuction with (7), $\beta_{\underline{v}} > \beta_{\overline{v}}$ must always hold.

Part (ii). Suppose instead that $\partial\beta_{\overline{v}}/\partial\kappa > 0$ when $\partial\beta_{\underline{v}}/\partial\kappa \leq 0$. Using (29), it follows that:

$$\frac{d\beta_{\overline{v}}}{d\kappa} < \frac{\beta_{\overline{v}}}{\beta_{\underline{v}}} \frac{d\beta_{\underline{v}}}{d\kappa} \leq 0;$$

which contradicts the fact that $\partial\beta_{\overline{v}}/\partial\kappa > 0$ when $\partial\beta_{\underline{v}}/\partial\kappa \leq 0$. As a result, if $\underline{v} \notin \mathbb{J}$, then $\partial\beta_{\overline{v}}/\partial\kappa \leq 0$ must hold. ■

Proof of Corollary 1.

Preliminarily, recall $\int_{z\in\mathbb{V}} \beta_z \, dz = 1$, which implies $\int_0^1 (\partial\beta_z/\partial\kappa)\, dz = 0$.[14]

Part (i). Claim immediately follows since, whenever $\kappa < \underline{\kappa}$, $\partial\beta_z/\partial\kappa = 0$ for all $z \in \mathbb{V}$.

Part (ii). Note first that when $\kappa \geq \underline{\kappa}$, the set $\mathbb{J} \neq \emptyset$. As a result, from Proposition 1, Part (i), it follows that $\int_0^v (\partial\beta_z/\partial\kappa)\, dz > \int_v^1 (\partial\beta_z/\partial\kappa)\, dz$. Then, since $\int_0^v (\partial\beta_z/\partial\kappa)\, dz + \int_v^1 (\partial\beta_z/\partial\kappa)\, dz = 0$, we must necessarily have that $\int_0^v (\partial\beta_z/\partial\kappa)\, dz > 0$. ■

D Auxiliary Derivations and Proofs

Proof of Claim 1

Recall that $q_v = 1$, $\forall v \notin \mathbb{L}$. For all other varieties, (22) in conjunction with (20) yield:

$$\eta(v) + \ln\alpha(v) + [\eta(v) - 1]\ln q_v = \eta(0) + \ln\alpha(0) + [\eta(0) - 1]\ln q_0, \ \forall v \in \mathbb{L}.$$

Isolating $[\eta(v) - 1]\ln q_v$, and applying exponentials to both sides gives:

$$(q_v)^{\eta(v)-1} = \frac{e^{\eta(0)}}{e^{\eta(v)}}\frac{\alpha(0)}{\alpha(v)}(q_0)^{\eta(0)-1}, \ \forall v \in \mathbb{L}.$$

Finally, raising both sides to the power $[\eta(v) - 1]^{-1}$, and considering Lemma 2, (25) obtains.

Proof of Claim 2

By definition of $\mathbb{L}$, $\lambda_{\tilde{v}(\kappa)} = 0$. Thus, the condition (22) applied on $\tilde{v}(\kappa)$ yields:

$$\eta(\tilde{v}(\kappa)) + \ln[\alpha(\tilde{v}(\kappa))/w] - \ln\kappa + \ln Q = -[\eta(\tilde{v}(\kappa)) - 1]\ln q_{\tilde{v}(\kappa)} \tag{30}$$

Suppose now that $q_{\tilde{v}(\kappa)} > 1$, and take some $\varepsilon \in (0, 1 - \tilde{v}(\kappa)]$. Then, since $v = \tilde{v}(\kappa) + \varepsilon \notin \mathbb{L}$, it must be the case that:

$$\eta(\tilde{v}(\kappa) + \varepsilon) + \ln[\alpha(\tilde{v}(\kappa) + \varepsilon)/w] - \ln\kappa + \ln Q = \lambda_{\tilde{v}(\kappa)+\varepsilon}. \tag{31}$$

Then, by continuity of $\eta(\cdot)$ and $\alpha(\cdot)$, and using the result in (30), we must have:

$$\lim_{\varepsilon\to 0}\{\eta(\tilde{v}(\kappa) + \varepsilon) + \ln[\alpha(\tilde{v}(\kappa) + \varepsilon)/w] - \ln\kappa + \ln Q\} = -[\eta(\tilde{v}(\kappa)) - 1]\ln q_{\tilde{v}(\kappa)} < 0.$$

[14] Note that it is then trivial to observe that $\partial\vartheta(1)/\partial\kappa = 0$, $\forall\kappa > 0$.

Hence, $q_{\tilde{v}(\kappa)} > 1$ cannot possibly hold when $\tilde{v}(\kappa) < 1$ as it would imply that $\lambda_{\tilde{v}(\kappa)+\varepsilon} < 0$ in (31) for $\varepsilon \to 0$, violating (20).

Proof of $\partial\vartheta(m)/\partial w \leq 0$.

Suppose first that $\tilde{v} < m$. Then, $\mathbb{L} \subset [0, m)$. Differentiating (22) with respect to w yields:

$$\frac{\eta(v) - 1}{q_v} \frac{\partial q_v}{\partial w} + \frac{1}{Q} \frac{\partial Q}{\partial w} = 0, \ \forall v \in \mathbb{L}. \tag{32}$$

Furthermore, from (25) it follows that:

$$\frac{\partial q_v}{\partial w} = \frac{\eta(0) - 1}{\eta(v) - 1} \frac{q_v}{q_0} \frac{\partial q_0}{\partial w}, \ \forall v \in \mathbb{L}. \tag{33}$$

Since $\partial Q/\partial w = \int_0^{\tilde{v}} (\partial q_z/\partial w)\, dz$, combining (32) and (33) yields:

$$\left(1 - \tilde{v} + \int_0^{\tilde{v}} \frac{\eta(z)}{\eta(z) - 1} q_z dz\right) \frac{\eta(0) - 1}{q_0} \frac{1}{Q} \frac{\partial q_0}{\partial w} = 0 \quad \Rightarrow \quad \frac{\partial q_0}{\partial w} = 0,$$

Therefore, using again (33), $\partial q_v/\partial w = 0$ for all $v \in [0, \tilde{v}]$ obtains. In addition, because of Lemma 2, it must thus be the case that $\partial q_v/\partial w = 0$ holds as well for all $v \in (\tilde{v}, 1]$. Finally, recalling (7) it then follows that $\partial\beta_v/\partial w = 0$ for all $v \in \mathbb{V}$, which in turn implies that $\partial\vartheta(m)/\partial w = 0$.

Suppose now that $\tilde{v} \geq m$. Differentiating (22) with respect to w now yields:

$$\frac{\eta(v) - 1}{q_v} \frac{\partial q_v}{\partial w} + \frac{1}{Q} \frac{\partial Q}{\partial w} = \begin{cases} 0, & \forall v \in [0, m) \\ 1/w, & \forall v \in [m, \tilde{v}] \end{cases} \tag{34}$$

From (34) it follows that a necessary condition for $\partial\vartheta(m)/\partial w > 0$ to hold is that $\partial Q/\partial w < 0$.[15] However, (34) means that if $\partial Q/\partial w < 0$, then $\partial q_v/\partial w > 0$ should hold for all $v \in [m, \tilde{v}]$. If $\tilde{v} = 1$, it must be straightforward to observe that $\partial Q/\partial w < 0$ cannot thus hold. Alternatively, if $\tilde{v} < 1$, then $\partial Q/\partial w < 0$ would require that $\partial q_v/\partial w < 0$ prevails for some $v \in (\tilde{v}, 1]$ which is not feasible either since it would lead to violating the constraint $q_v \leq 1$. As a result, $\partial Q/\partial w > 0$ must hold, which in turn implies $\partial\vartheta(m)/\partial w < 0$. ■

Proof of $\partial\vartheta^*(m)/\partial w \leq 0$.

Suppose first that $\tilde{v}^* < m$. Then, $\mathbb{L}^* \subset [0, m)$. Differentiating (22) – adjusted for representing an individual from the South – with respect to w yields:

$$\frac{\eta(v) - 1}{q_v^*} \frac{\partial q_v^*}{\partial w} + \frac{1}{Q^*} \frac{\partial Q^*}{\partial w} = -\frac{1}{w}, \ \forall v \in \mathbb{L}^*. \tag{35}$$

[15] Otherwise, if $\partial Q/\partial w \geq 0$, (34) would imply that $\partial q_v/\partial w \leq 0$ for all $v \in [0, m)$. Recalling (7), it is then straightforward to observe that $\partial Q/\partial w \geq 0$ would mean $\partial\beta_v/\partial w \leq 0$ for all $v \in [0, m)$, which in turn leads to $\partial\vartheta(m)/\partial m \leq 0$.

In addition, from (25) it follows that:

$$\frac{\partial q_v^*}{\partial w} = \frac{\eta(0) - 1}{\eta(v) - 1} \frac{q_v^*}{q_0^*} \frac{\partial q_0^*}{\partial w}, \ \forall v \in \mathbb{L}^*. \tag{36}$$

Combining (35) and (36) leads to:

$$\left(1 - \tilde{v}^* + \int_0^{\tilde{v}^*} \frac{\eta(z)}{\eta(z) - 1} q_z dz\right) \frac{\eta(0) - 1}{q_0^*} \frac{1}{Q^*} \frac{\partial q_0^*}{\partial w} = -\frac{1}{w} \quad \Rightarrow \quad \frac{\partial q_0^*}{\partial w} < 0.$$

Hence, using again (36), $\partial q_v^* / \partial w < 0$ for all $v \in [0, \tilde{v}^*]$ obtains, which in turn implies $\partial Q^* / \partial w < 0$. Next, since for all $v \geq \tilde{v}^*$ the constraint $q_v^* \geq 1$ is binding, it must be the case that $\partial q_v^* / \partial w \geq 0$, $\forall v \in (\tilde{v}^*, 1]$. As a result, because of (7), $\partial \beta_v^* / \partial w > 0$ for all $v \in [m, 1]$ follows, which in turn implies $\partial \vartheta^*(m) / \partial w < 0$.

Suppose now $\tilde{v}^* \geq m$. Differentiating (22) with respect to w now yields:

$$\frac{\eta(v) - 1}{q_v^*} \frac{\partial q_v^*}{\partial w} + \frac{1}{Q^*} \frac{\partial Q^*}{\partial w} = \begin{cases} -1/w, & \forall v \in [0, m) \\ 0, & \forall v \in [m, \tilde{v}^*] \end{cases} \tag{37}$$

If $\partial Q^* / \partial w \geq 0$, from (37) it follows that $\partial q_v^* / \partial w < 0$ for all $v \in [0, m)$, which using (7) entails that $\partial \beta_v^* / \partial w < 0$ for all $v \in [0, m)$ too. As a consequence, a necessary condition for $\partial \vartheta^*(m) / \partial w > 0$ to hold is that $\partial Q^* / \partial w < 0$. However, notice that if $\partial Q^* / \partial w < 0$, then (37) implies $\partial q_v^* / \partial w > 0$ for all $v \in [m, \tilde{v}^*]$. Furthermore, in case $\tilde{v}^* < 1$, since $\forall v \in (\tilde{v}^*, 1]$ the constraint $q_v^* \geq 1$ is binding, $\partial q_v^* / \partial w \geq 0$ must necessarily hold for all $v \in (\tilde{v}^*, 1]$. As a result, if $\partial Q^* / \partial w < 0$, then $\partial \beta_v^* / \partial w > 0$ for all $v \in [m, 1]$, which in turn leads to $\partial \vartheta^*(m) / \partial w < 0$. ■

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